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Abstract

The hypothesis that the fourth moments of the field amplitude distribution are related to the second moments as in a normal distribution is examined to determine whether it is consistent with the equations of motion for stationary, homogeneous, and incompressible hydromagnetic turbulence. An application of this hypothesis to the two-point, two-time amplitude distribution of the velocity field and magnetic field leads to a gross violation of energy conservation at high Reynolds number. This result holds also in the non-magnetic case. An estimate of the magnitude of the violation yields a rate of energy generation of order $E_i = E_i/\tau_i$, where E_i is the energy contained in the inertial range and τ_i is a period characteristic of modes in the inertial range. The results obtained are discussed in relation to a recent theory of turbulence formulated by Chandrasekhar.

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1. Introduction

A number of workers in turbulence theory have employed the hypothesis that certain fourth moments of the statistical distribution of the field amplitudes are related to second moments as in a normal distribution. This hypothesis apparently was first employed by Millionshtchikov[1]. Later, it was applied to the two-point, simultaneous velocity distribution by Heisenberg[2], Obukhoff[3], Batchelor[4], Chandrasekhar[5], and others, in order to estimate pressure fluctuations in turbulent flows. Heisenberg[2] used an implicit application of the hypothesis to the two-time velocity distribution in an attempt to develop equations determining the spectrum of high Reynolds number turbulence, and, more recently, a theory of turbulence dynamics employing this form of the hypothesis has been formulated by Chandrasekhar[6] and extended to hydromagnetic turbulence[7]. Proudman and Reid[8] had previously applied the hypothesis of normal relationship to the three-point, simultaneous velocity distribution in a treatment of the decay of homogeneous turbulence. Some experimental evidence with regard to these hypotheses is discussed by Uberoi[9] and Batchelor[10].

In the present paper, the consistency of the hypothesis of normal relationship of fourth and second moments is examined for the case of stationary, homogeneous, incompressible hydromagnetic turbulence. It is concluded that this hypothesis applied to the two-time, two-point distribution of the field amplitudes leads to gross violation of energy conservation at high Reynolds numbers. The conclusions reached are discussed in relation to the dynamical theory of Chandrasekhar, mentioned above.

2. The equations of motion

Consider an incompressible fluid of uniform density ρ, kinematic viscosity ν, permeability μ, and conductivity σ, which is subjected to an external force field $\vec{F}(\vec{x},t)$. Under the related assumptions of nonrelativistic motion, negligible displacement current, and high conductivity[11], one may derive from Maxwell's equations, Ohm's law, and conservation of the total stress-energy tensor the divergence conditions

$$\vec{\nabla}\cdot\vec{B} = \vec{\nabla}\cdot\vec{u} = 0 \tag{2.1}$$

and the equations of motion

$$\dot{\vec{B}} - \vec{\nabla} \times (\vec{u}\times\vec{B}) - (4\pi\mu\sigma)^{-1}\, c^2\, \nabla^2\, \vec{B} = 0 \tag{2.2}$$

$$\dot{\vec{u}} + (\vec{u}\cdot\vec{\nabla})\vec{u} - (4\pi\mu\rho)^{-1}\, (\vec{\nabla}\times\vec{B}) \times \vec{B} + \rho^{-1}\, \vec{\nabla}p - \nu\nabla^2\, \vec{u} = \vec{F}, \tag{2.3}$$

where $\vec{u}(\vec{x},t)$ is the velocity, $\vec{B}(\vec{x},t)$ is the magnetic induction, and $p(\vec{x},t)$ is the pressure.

Equations (2.2), (2.3) may be solved for p and rewritten in the tensor form

$$\dot{w}_i - \bar{\nu}\nabla^2 w_i = \nabla_j(u_i w_j - u_j w_i) \tag{2.4}$$

$$\dot{u}_i - \nu\nabla^2 u_i = P_{ij}(\vec{\nabla})\left[\nabla_\ell(w_j w_\ell - u_j u_\ell) + F_j\right], \tag{2.5}$$

where

$$\vec{w} = (4\pi u\rho)^{-1/2}\vec{B}, \qquad \bar{\nu} = (4\pi\mu\sigma)^{-1}\, c^2, \tag{2.6}$$

and

$$P_{ij}(\vec{\nabla}) = \delta_{ij} - \nabla^{-2}\,\nabla_i\nabla_j \qquad \left[\nabla_i = \partial/\partial\, x^i\right]. \tag{2.7}$$

The projection operator $P_{ij}(\nabla)$ applied to the nonsolenoidal part of a vector field gives zero; it represents the action of the pressure forces. Both divergence conditions (2.1) are preserved by the equations of motion.

The total energy of the system is

$$\text{Energy} = \frac{1}{2}\rho\int\left(u^2 + w^2\right)d^3x. \tag{2.8}$$

If the flow is of finite extent, or obeys cyclic boundary conditions, one may verify by partial integration the relation

$$\int\left[w_i\nabla_j(u_i w_j - u_j w_i) + u_i P_{ij}(\nabla)\nabla_\ell(w_j w_\ell - u_j u_\ell)\right]d^3x = 0, \tag{2.9}$$

which demonstrates the conservative nature of the nonlinear interaction.

It is convenient in what follows to express the field amplitudes in Fourier series, employing the artifice of cyclic boundary conditions on the faces of a cube. An equivalent analysis could be carried out in terms of Fourier integrals, but at the expense of more cumbersome notation. The equations of motion for the Fourier coefficients associated with wave vector $\vec{k}$ are

$$k_i w_i(\vec{k}) = k_i u_i(\vec{k}) = 0 \tag{2.10}$$

$$\dot{w}_i(\vec{k}) + \bar{\nu}k^2 w_i(\vec{k}) = ik_j \sum_{\vec{k}'} \left[u_i(\vec{k} - \vec{k}')w_j(\vec{k}') - u_j(\vec{k} - \vec{k}')w_i(\vec{k}')\right] \tag{2.11}$$

$$\dot{u}_i(\vec{k}) + \nu k^2 u_i(\vec{k}) = ik_\ell P_{ij}(\vec{k}) \sum_{\vec{k}'} \left[w_j(\vec{k} - \vec{k}')w_\ell(\vec{k}') - u_j(\vec{k} - \vec{k}')u_\ell(\vec{k}')\right] + P_{ij}(\vec{k})F_j(\vec{k}), \tag{2.12}$$

and the energy per unit mass is

$$E = \frac{1}{2}\sum_{\vec{k}}\left[u_i^*(\vec{k})u_i(\vec{k}) + w_i^*(\vec{k})w_i(\vec{k})\right]. \tag{2.13}$$

Let the linearly independent real and imaginary parts of the independent vector components of all the Fourier coefficients $\vec{u}(\vec{k})$, $\vec{w}(\vec{k})$ be arranged in some one-dimensional serial order and be denoted by $q_\alpha/\sqrt{2}$, $\alpha = 1,2....$ For each allowed wave vector $\vec{k}$ there are two independent vector components of each of the Fourier coefficients of the real, solenoidal vector fields $\vec{u}$ and $\vec{w}$; thus, there are eight q's for each allowed pair $\vec{k}$, $-\vec{k}$.

The equations of motion of the q's may be written

$$\dot{q}_\alpha + \nu_\alpha q_\alpha = \sum_{\beta,\gamma} A_{\alpha\beta\gamma} q_\beta q_\gamma + f_\alpha \qquad \left[A_{\alpha\beta\gamma} = A_{\alpha\gamma\beta}\right], \tag{2.14}$$

where the coefficients $A_{\alpha\beta\gamma}$ and the damping factors ν_α contain the $\vec{k}$'s implicitly and the f_α are the suitably ordered and normalized real and imaginary parts of the independent vector components of the solenoidal force field $P_{ij}(\vec{\nabla})F_j$. All the quantities appearing in (2.14) are real, and the ν_α are all positive. (No summation convention is employed for the Greek indices.)

The total energy per unit mass is given by

$$E = \frac{1}{2}\sum_\alpha (q_\alpha)^2, \tag{2.15}$$

and equation (2.9), which expresses conservation of energy by the nonlinear interaction, implies

$$\sum_{\alpha,\beta,\gamma} A_{\alpha\beta\gamma} q_\alpha q_\beta q_\gamma = 0. \tag{2.16}$$

This relation must hold for all values of the independent variables q. Taking all the q's to be zero except q_μ, q_λ, q_σ, one finds

$$A_{\mu\lambda\sigma} + A_{\lambda\sigma\mu} + A_{\sigma\mu\lambda} = 0. \tag{2.17}$$

If the three-mode interaction of q_μ, q_λ, q_σ is defined by those nonlinear terms in the equations of motion of these three variables which contain no other q's, it is apparent from (2.14), (2.17) that this elementary interaction contributes a vanishing part to the time-rate of change of E and is therefore individually conservative.

The selection rules to determine which A's are nonvanishing are fixed by the orthogonality conditions (2.1) and by which wave vector pairs contribute to the convolution sums in equations (2.11), (2.12). It will be assumed that there is no mean flow and no uniform component of magnetic field. Then the Fourier coefficients $\vec{u}(0)$, $\vec{w}(0)$ vanish, and the orthogonality conditions permit only pairs of Fourier coefficients such that $\vec{k}$ and $\vec{k} - \vec{k}'$ are nonparallel to contribute to the convolution sums. In this case, the selection rules on the A's include the condition

$$A_{\mu\lambda\sigma} = 0, \quad \text{unless } \mu, \lambda, \sigma \text{ are all different.} \tag{2.18}$$

If the Fourier coefficient $\vec{F}(0)$ vanishes, the conditions $\vec{u}(0)$, $\vec{w}(0) = 0$ are preserved by the equations of motion.

The various equations of motion (2.4), (2.5), (2.11), (2.12), (2.14) may be integrated with respect to time to yield integral equations. From (2.5) and (2.14) one obtains, respectively,

$$u_i = \int_{-\infty}^{t} P_{ij}(\nabla)\exp[\nu(t-t')\nabla^2]\left[\nabla_\ell(w_j'w_\ell'-u_j'u_\ell') + F_j'\right]dt' \tag{2.19}$$

$$q_\alpha = \int_{-\infty}^{t} \exp[-\nu(t-t')]\left[\sum_{\beta,\gamma} A_{\alpha\beta\gamma}q_\beta'q_\gamma' + f_\alpha'\right]dt', \tag{2.20}$$

where unprimed variables have argument t, primed variables have argument t', and the boundary condition of finiteness at $t = -\infty$ has been applied.

3. The statistical equations of homogeneous stationary turbulence

The statistical theory of turbulence is concerned with distributions of solutions of the equations of motion in a function space of space-time functions.

It is usually presumed, on physical grounds, that the finite-order moments of the distribution and their finite-order derivatives exist. Stationary, homogeneous, or isotropic distributions are defined by the invariance of the moments under time displacement, space displacement, or proper rotations, respectively.

In the stationary, homogeneous case to be considered here, one may define the moments

$$
(3.1)\quad \left\{\begin{aligned}
&C_{ij}(\vec{y},\tau) \equiv \left\langle u_i(\vec{x},t)u_j(\vec{x}',t')\right\rangle \\
&\dot{C}_{ij}(\vec{y},\tau) \equiv \left\langle u_i(\vec{x},t)u_j(\vec{x}',t')\right\rangle \\
&D_{ijk}(\vec{y},\tau) \equiv \left\langle u_i(\vec{x},t)u_j(\vec{x},t)u_k(\vec{x}',t')\right\rangle \\
&E_{ijk\ell}(\vec{y},\tau) \equiv \left\langle u_i(\vec{x},t)u_j(\vec{x},t)u_k(\vec{x}',t')u_\ell(\vec{x},t')\right\rangle \\
&G_{ij}(\vec{y},\tau) \equiv \left\langle F_i(\vec{x},t)u_j(\vec{x}',t')\right\rangle \\
&H_{ij\ell}(\vec{y},\tau) \equiv \left\langle u_i(\vec{x},t)u_j(\vec{x},t)F_\ell(\vec{x}',t')\right\rangle \\
&\qquad [\vec{y} = \vec{x} - \vec{x}', \quad \tau = t - t'] ,
\end{aligned}\right.
$$

which belong to the two-point, two-time distribution of $\vec{u}$ and $\vec{F}$; similar moments involving $\vec{w}$ may also be defined. It follows from the definitions that $C_{ij}(\vec{y},\tau) = C_{ji}(-\vec{y},-\tau)$, with analogous relations for other moments.

Multiplying equation (2.5) for $\dot{u}_i(\vec{x},t)$ by $u_k(\vec{x}',t')$ and averaging over the distribution, one obtains, for the special case $\vec{w} = 0$, the statistical equation of motion

$$
(3.2)\qquad \dot{C}_{ik} - \nu\nabla^2 C_{ik} = P_{ij}(\vec{\nabla})[-\nabla_\ell D_{j\ell k} + G_{jk}],
$$

where $\vec{\nabla}$ operates with respect to $\vec{y}$. If $u_k(\vec{x}',t')$ is expressed according to the integral equation (2.19) before the averaging, one can obtain, instead, the integro-differential equation

$$\dot{C}_{ik}(\vec{y},0) - \nu\nabla^2 C_{ik}(\vec{y},0) = \int_0^\infty P_{ij}(\vec{\nabla})\nabla_\ell \exp(\nu\tau\nabla^2) \tag{3.3}$$
$$\times \left[- P_{km}(\vec{\nabla})\nabla_n E_{j\ell mn}(\vec{y},-\tau) + H_{j\ell k}(\vec{y},-\tau)\right]d\tau + P_{ij}(\vec{\nabla})G_{jk}(\vec{y},0),$$

in which the fourth moments $E_{j\ell mn}$ appear instead of the third moments $D_{j\ell k}$.

The moments

$$R_{\mu\lambda}(\tau) \equiv \langle q_\mu(t)q_\lambda(t')\rangle$$
$$\dot{R}_{\mu\lambda}(\tau) \equiv \langle \dot{q}_\mu(t)q_\lambda(t')\rangle \tag{3.4}$$
$$S_{\mu\lambda\alpha}(\tau) \equiv \langle q_\mu(t)\, q_\lambda(t)q_\alpha(t')\rangle$$
$$T_{\mu\lambda\alpha\beta}(\tau) \equiv \langle q_\mu(t)q_\lambda(t)q_\alpha(t')q_\beta(t')\rangle$$

describe the two-time q-distribution. The definitions require $R_{\mu\lambda}(\tau) = R_{\lambda\mu}(-\tau)$ with analogous relations for the other moments. In analogy to equations (3.2), (3.3), one may derive the statistical equations of motion

$$\dot{R}_{\alpha\mu} + \nu_\alpha R_{\alpha\mu} = \sum_{\beta\gamma} A_{\alpha\beta\gamma} S_{\beta\gamma\mu} \tag{3.5}$$

$$\dot{R}_{\alpha\mu}(0) + \nu_\alpha R_{\alpha\mu}(0) = \sum_{\beta,\gamma,\lambda,\sigma} A_{\alpha\beta\gamma}A_{\mu\lambda\sigma}\int_0^\infty e^{-\nu_\alpha\tau}T_{\lambda\sigma\beta\gamma}(\tau)d\tau, \tag{3.6}$$

valid for modes such that $f_\alpha = 0$.

The quantity $\dot{R}_{\alpha\alpha}(0) = \langle \dot{q}_\alpha q_\alpha\rangle$ represents the average rate of increase of the energy of the mode q_α and must vanish in a stationary distribution. It is apparent from (3.5) that the contribution to $\langle \dot{q}_\alpha q_\alpha\rangle$ from the elementary three-mode interaction of distinct modes q_α, q_β, q_γ is

$$\Omega_{\alpha\beta\gamma} \equiv 2A_{\alpha\beta\gamma}\, S_{\beta\gamma\alpha}(0) = 2A_{\alpha\beta\gamma}\langle \dot{q}_\alpha q_\beta q_\gamma\rangle. \tag{3.7}$$

The individual conservation property of the three-mode interactions expressed by equations (2.17) has the consequence

$$\Omega_{\alpha\beta\gamma} + \Omega_{\beta\gamma\alpha} + \Omega_{\gamma\alpha\beta} = 0. \tag{3.8}$$

For fields $\vec{u}$ and $\vec{w}$ obeying cyclic boundary conditions on the faces of a cube of side L, the requirement of invariance of the statistical distribution under spatial displacement modulo L implies that the Fourier coefficients associated with different wave vector pairs $\vec{k}$, $-\vec{k}$ must have zero correlation and that the real parts of the coefficients for a given wave-number pair must be uncorrelated with the imaginary parts. In addition, it is always possible, in this case, to choose principal axes for each coefficient so that the two independent vector components of $\vec{u}(\vec{k})$ are uncorrelated and the two independent components of $\vec{w}(\vec{k})$ are uncorrelated*. The equations of motion (2.4), (2.5) are invariant under sign reversal of $\vec{w}$. This ensures the existence of solution distributions in which $\vec{u}$ and $\vec{w}$ are uncorrelated, as has been noted by Chandrasekhar[7]. Since the q's are the independent vector components of the real and imaginary parts of the Fourier coefficients $\vec{u}(\vec{k})$, $\vec{w}(\vec{k})$, it follows that for such distributions the requirement of homogeneity and the choice of principal axes described yields the conditions

$$R_{\alpha\beta}(\tau) = \delta_{\alpha\beta} R_{\alpha\alpha}(\tau). \tag{3.9}$$

A two-time distribution will be called <u>quasi-normal</u> if for every choice of four quantities a, b, c', d' (where no prime denotes time t and a prime denotes time t') the fourth and second moments are related by the equation

$$\langle abc'd' \rangle = \langle ab \rangle \langle c'd' \rangle + \langle ac' \rangle \langle bd' \rangle + \langle ad' \rangle \langle bc' \rangle , \tag{3.10}$$

which holds for a normal distribution. The hypothesis that the two-time stationary distribution of the q's is quasi-normal implies

$$T_{\mu\lambda\alpha\beta}(\tau) = R_{\mu\lambda}(0) R_{\alpha\beta}(0) + R_{\mu\alpha}(\tau) R_{\lambda\beta}(\tau) + R_{\mu\beta}(\tau) R_{\lambda\alpha}(\tau). \tag{3.11}$$

If $A = \sum \alpha_\mu q_\mu$, $B = \sum \beta_\mu q_\mu$, $C' = \sum \gamma_\mu q'_\mu$, $D' = \sum \delta_\mu q'_\mu$ are any four linear combinations of the q's with coefficients constant over the distribution, equation (3.11) yields

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*For isotropic turbulence, this is true independently of the choice of axes. Symmetry conditions on the correlation and spectrum tensors in homogeneous turbulence are discussed in [10], Chapter III.

$$(3.12)\quad \langle ABC'D'\rangle = \sum \alpha_\mu \beta_\lambda \gamma_\sigma \delta_\tau \langle q_\mu q_\lambda q'_\sigma q'_\tau \rangle$$

$$= \sum \alpha_\mu \beta_\lambda \gamma_\sigma \delta_\tau \left[\langle q_\mu q_\lambda \rangle \langle q'_\sigma q'_\tau \rangle + \langle q_\mu q'_\sigma \rangle \langle q_\mu q'_\tau \rangle + \langle q_\lambda q'_\tau \rangle \langle q_\lambda q'_\sigma \rangle \right]$$

$$= \langle AB \rangle \langle C'D' \rangle + \langle AC' \rangle \langle BD' \rangle + \langle AD' \rangle \langle BC' \rangle \quad .$$

Thus, since the q's are the real and imaginary parts of all the Fourier coefficients of $\vec{u}$ and $\vec{w}$, equation (3.11) implies that the four-point, two-time joint distribution of $\vec{u}$ and $\vec{w}$ is quasi-normal.

The hypothesis that the two-point, two-time joint distribution of $\vec{u}$ and $\vec{w}$ is quasi-normal is evidently considerably less restrictive than equation (3.11). However, it is important, in what follows, to note that if any two-point, two-time fourth moment of $\vec{u}$ and $\vec{w}$ is reduced to second moments by the two-point quasi-normality hypothesis, and then expressed in terms of the $R_{\alpha\beta}$ by Fourier analysis, the result must be identical with that which would be obtained by applying (3.11) without restriction. This may be seen by considering $\vec{u}$ and $\vec{w}$ as sums over the q's and employing (3.12) in reverse. The hypothesis that the two-time, two-point joint distribution of $\vec{u}$ and $\vec{w}$ is quasi-normal implies

$$(3.13)\quad E_{ijk\ell}(\vec{y},\tau) = C_{ij}(0,0)C_{k\ell}(0,0) + C_{ik}(\vec{y},\tau)C_{j\ell}(\vec{y},\tau)$$

$$+ C_{i\ell}(\vec{y},\tau)C_{jk}(\vec{y},\tau),$$

with similar conditions for moments involving $\vec{w}$.

4. Consequences of the quasi-normality hypothesis

It may be seen very simply that the quasi-normality hypothesis (3.11) is inconsistent with the conservation properties of the nonlinear interaction. Consider any three modes q_α, q_β, q_γ such that f_α, f_β, f_γ vanish. They could be magnetic modes or velocity modes in the inertial or dissipation ranges. Substituting the integral expression (2.20) for q_α in (3.7), one finds for the mean rate of transfer of energy to mode q_α through the three-mode interaction with q_β, q_γ,

$$(4.1)\quad \Omega_{\alpha\beta\gamma} = 2\, A_{\alpha\beta\gamma} \sum_{\mu,\lambda} A_{\alpha\mu\lambda} \int_0^\infty e^{-\nu_\alpha \tau}\, T_{\mu\lambda\beta\gamma}(\tau)\, d\tau .$$

If the hypothesis (3.11) is asserted, (4.1) may be rewritten*

$$(4.2)\quad \Omega_{\alpha\beta\gamma} = 4(A_{\alpha\beta\gamma})^2 \int_0^\infty e^{-\nu_\alpha \tau} R_{\beta\beta}(\tau) R_{\gamma\gamma}(\tau)\, d\tau ,$$

where the relations (2.18), (3.9) have been used.

It may now be noted that the integrals on the right side of (4.2) are all positive quantities. This follows from the fact that the spectrum of $e^{-\nu_\alpha|\tau|}$ is everywhere positive and the spectra of the correlation functions $R_{\beta\beta}(\tau)$, $R_{\gamma\gamma}(\tau)$ cannot be negative anywhere. Since the spectrum of a product of functions with non-negative spectra has itself a non-negative spectrum, and since the integral of the even function $e^{-\nu_\alpha|\tau|} R_{\beta\beta}(\tau) R_{\gamma\gamma}(\tau)$ gives the value at the origin of its Fourier transform, none of the integrals can be negative. Given $\nu_\alpha \neq 0$ it follows further that none of the integrals can be zero, if $R_{\beta\beta}(\tau)$, $R_{\gamma\gamma}(\tau)$ are well behaved. This may be seen by noting that $R_{\beta\beta}(0)$, $R_{\gamma\gamma}(0) > 0$, so that if $e^{-\nu_\alpha|\tau|}$ is replaced by a new function in which the cusp is rounded off arbitrarily close to the origin, but whose spectrum is still everywhere non-negative, the values of the integrals are reduced but must still be non-negative. If one takes $\nu_\alpha = 0$, a given integral can vanish only if the corresponding pair of modes have no frequencies in common over the distribution, that is, only if there are almost no pairs of solutions in the distribution such that q_β of one has frequencies in common with q_γ of the other.

Applying the same procedure and argument to $\Omega_{\beta\gamma\alpha}$ and $\Omega_{\gamma\alpha\beta}$, we obtain

$$(4.3)\quad \Omega_{\alpha\beta\gamma} > 0, \quad \Omega_{\beta\gamma\alpha} > 0, \quad \Omega_{\gamma\alpha\beta} > 0,$$

which violates the detailed conservation condition (3.8).

Certain consequences of the two-point, two-time quasi-normality hypothesis will be discussed now. Consideration will be confined to the nonmagnetic case $\vec{w} = 0$. This results in appreciable reduction in manipulations, and the generalization to the magnetic case (with appropriate augmentation of equation (3.13)) is straightforward.

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*The use of integrated equations of motion to relate fourth and second moments apparently was first described by W. Heisenberg[2].

The decomposition of (3.3) into Fourier series yields an expression for the Fourier coefficients of $\dot{C}_{ik}(\vec{y},0)$ in terms of the Fourier coefficients of the tensors $\mathcal{C}, \mathcal{E}, \mathcal{G}, \mathcal{H}$. Consider a wave vector $\vec{k}$ such that the Fourier coefficient $F(\vec{k})$ of the driving force vanishes. The Fourier coefficients of $\mathcal{G}$ and $\mathcal{H}$ are cross-power-spectra of $\vec{F}(\vec{x},t)$ with functions of $\vec{u}(\vec{x},t)$ and must vanish for such $\vec{k}$. The Fourier coefficients of $\dot{C}_{ik}(\vec{y},0)$ are $\langle u_i^*(\vec{k})u_k(\vec{k})\rangle$ and determine $\dot{R}_{\alpha\alpha}(0)$, if q_α is a velocity mode belonging to wave vector $\vec{k}$. It follows that equation (3.3) permits us to express $\dot{R}_{\alpha\alpha}(0)$ in terms of Fourier coefficients of $\mathcal{C}$ and $\mathcal{E}$ alone if q_α is a velocity mode associated with a wave number for which $F(\vec{k}) = 0$.

If, now, the two-point, two-time quasi-normality hypothesis (3.13) is asserted, the tensor $\mathcal{E}$ may be expressed in terms of the tensor $\mathcal{C}$, and, in consequence, the Fourier coefficients of $\mathcal{E}$ may be expressed in terms of sums of products of Fourier coefficients of $\mathcal{C}$ and, in turn, expressed in terms of the quantities $R_{\alpha\beta}$. According to the discussion in the preceding section, this expression in terms of the $R_{\alpha\beta}$ must be identical with that obtained by applying the stronger quasi-normality hypothesis (3.11) to the equations of motion. Applying (3.11) to (3.6) and specializing to the case $\alpha = \mu$, one finds with the aid of the relations (2.18), (3.9),

$$\dot{R}_{\alpha\alpha}(0) = 2\sum_{\beta,\gamma}(A_{\alpha\beta\gamma})^2\int_0^\infty e^{-\nu_\alpha\tau}R_{\beta\beta}(\tau)R_{\gamma\gamma}(\tau)\,d\tau - \nu_\alpha R_{\alpha\alpha}(0). \tag{4.4}$$

The quantity $\dot{R}_{\alpha\alpha}(0) = \langle\dot{q}_\alpha q_\alpha\rangle$ is the average rate of increase of the energy of the mode q_α and must vanish in stationary turbulence. The vanishing of the right side of (4.4) is the condition of energy balance between the net energy input due to nonlinear interaction and the viscous dissipation for a mode not subjected to external driving forces.

It is not immediately apparent that (4.4) is impossible to satisfy, but the relation may easily be seen to indicate behavior at variance with the usual physical picture of the dynamics of the energy cascade process in high Reynolds number turbulence. Consider a mode in the inertial range. It is accepted that the dynamics of such modes are almost entirely controlled by the Reynolds stresses and pressure forces and that the direct effect of viscosity in this range is negligible. If one now imagines reducing the damping factor ν_α while leaving undisturbed the damping factors of the other modes, the effect on the excitation of q_α and on the dynamics of the other modes should be negligible. According to (4.4), however, if

$\dot{R}_{\alpha\alpha}(0)$ is to remain zero the effect of reducing ν_α by, say, a factor of 10 must be profound. Either the mean excitation $R_{\alpha\alpha}(0)$ of the mode q_α would have to increase by a large factor, or the positive integrals on the right would have to change significantly.

In any individual solution representing a turbulent flow at high Reynolds number, the time variation of a mode q_α in the inertial range should be exceedingly complicated and its frequency spectrum should be broad. In a flow not subjected to random external perturbations it is perhaps conceivable that the frequency spectra of modes in the inertial range could have some rapidly oscillating structure, but in an actual flow such oscillations surely would be smeared out as a result of perturbations. It follows that in a distribution representing the presumed ergodic behavior of high Reynolds number turbulence, the averages $R_{\beta\beta}(\tau)$, $R_{\gamma\gamma}(\tau)$ should be smooth well-behaved functions of τ. Further, it seems likely that these functions should not exhibit significant regions of negativity. This comment is based on the observation that the forces $A_{\beta\mu\lambda}q_\mu q_\lambda$ exerted on mode q_β through the three-mode interactions are quadratic in the q's, so that their frequency spectra, and consequently the frequency spectrum of q_β should contain strong zero frequency components; hence the moments $\int_0^\infty R_{\beta\beta}(\tau)\, d\tau$ should not differ greatly in order of magnitude from $\tau_\beta R_{\beta\beta}(0)$, where τ_β is some characteristic period for the mode.

If $R_{\alpha\alpha}(\tau)$, $R_{\beta\beta}(\tau)$, $R_{\gamma\gamma}(\tau)$ in (4.4) are such smooth functions, it is difficult to see how they could adjust to a large reduction of ν_α without significant change of the excitation levels or the characteristic mode periods. In the next section an estimate will be made of the magnitude of the sum over integrals in equation (4.4) for high Reynolds number.

5. Discussion of a nondissipative system

An insight into the relation of the quasi-normality hypotheses to conservation of energy by the nonlinear interaction is provided by the study of an idealized flow system in which both dissipation and external forces are assumed to vanish. As in the last section, explicit consideration will be confined to the nonmagnetic case, $\vec{w} = 0$, the extension to the more general case being easy but more cumbersome.

If ν, $\vec{F}$ and $\vec{w}$ are taken equal to zero, it is well known that the solutions of (2.5) develop surfaces of discontinuity which correspond physically to shock fronts. It is not possible, for this reason, tc speak meaningfully of nontrivial stationary distributions of solutions of the inviscid equations. To avoid such behavior, the altered equation of motion

(5.1) $$\dot{u}_i = -\eta(K^{-2}\nabla^2)P_{ij}(\vec{\nabla})\nabla_\ell(u_j u_\ell)$$

will be adopted. The operator η, defined by

(5.2) $$\eta(S) = 1, \ |S| \leq 1, \ \eta(S) = 0, \ |S| > 1,$$

prevents the development of infinite gradients. It may be verified by application of Green's theorem that the property of conservation of the energy expression (2.8) is not altered with arbitrary choice of the operator η.

In terms of the Fourier representation, η provides a cut-off in wave number, suppressing completely all modes of wave vector $|\vec{k}| > K$. It may be regarded "physically" as representing an infinite viscosity for these modes*. The equations of motion for the finite number of modes which remain are

(5.3) $$\dot{q}_\alpha = \sum_{\beta,\gamma} A_{\alpha\beta\gamma} q_\beta q_\gamma \quad [\alpha = 1, 2, \ldots N],$$

where N is the total number of q's. The A's may be taken as identical with those in (2.14) if all A's referring to wave numbers $|\vec{k}| > K$ and to magnetic degrees of freedom are removed. The detailed conservation property (2.17) provides another demonstration that the operator η does not destroy energy conservation.

It is evident from equations (5.3), (2.18) that $\partial\dot{q}_\alpha/\partial q_\alpha = 0$, so that the new system obeys the Liouville equation

(5.4) $$\sum_{\alpha=1}^{N} \frac{\partial \dot{q}_\alpha}{\partial q_\alpha} = 0.$$

An immediate consequence is that any distribution of simultaneous amplitudes which depends only on the constant of motion $E = \frac{1}{2}\sum_\alpha (q_\alpha)^2$ is time invariant under the equations of motion; for such distributions there is equipartition of energy among the modes. In particular, the Maxwell distribution of simultaneous amplitudes

(5.5) $$Z \exp\left[-\beta \sum_\alpha (q_\alpha)^2\right],$$

where Z is a normalization constant and β is related to the mean energy per mode,

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*Application of a similar cut-off to an inviscid, adiabatically compressible fluid is discussed in [12].

is time invariant.

If the hypothesis that the two-point, two-time distribution of $\vec{u}$ is quasi-normal is now asserted, the procedures of the last section lead to the equation

$$(5.6) \qquad \dot{E} = \sum \langle \dot{q}_\alpha q_\alpha \rangle = 2 \sum_{\alpha,\beta,\gamma} (A_{\alpha\beta\gamma})^2 \int_0^\infty R_{\beta\beta}(\tau) R_{\gamma\gamma}(\tau) d\tau .$$

Energy conservation requires that $\dot{E}$ vanish. As discussed before, this would be possible only if, given q_β and q_γ such that $A_{\alpha\beta\gamma} \neq 0$, (almost) no functions q_β associated with any given solutions in the distribution should have any frequencies in common with the functions q_γ associated with (almost) any other solutions in the distribution. This would be absurd*.

It is possible to obtain a crude estimate of $\dot{E}$ in (5.6) if one adopts the simultaneous amplitude distribution (5.5) and assumes that the distribution averages $R_{\alpha\alpha}(\tau)$ are smooth, essentially positive functions for N large. Assume the modes are coupled in such a way that all the characteristic mode periods τ_α are of the same order of magnitude $\bar{\tau}$. Let the mean energy of the equally excited modes be $\frac{1}{2}R_{\alpha\alpha}(0) = \frac{1}{2}R$. Let some average value of the non-vanishing coefficients $A_{\alpha\beta\gamma}$ be A and let the average number of non-vanishing $A_{\alpha\beta\gamma}$ for fixed α be M. Then an estimate of $\vec{\tau}$ is given by

$$(5.7) \qquad \bar{\tau}^{-2} \sim R^{-1} \langle \dot{q}_\alpha \dot{q}_\alpha \rangle = R^{-1} \sum_{\beta,\gamma,\mu,\lambda} A_{\alpha\beta\gamma} A_{\alpha\mu\lambda} T_{\beta\gamma\mu\lambda}(0)$$

$$= 2 R^{-1} \sum_{\beta,\gamma} (A_{\alpha\beta\gamma})^2 R_{\beta\beta}(0) R_{\gamma\gamma}(0) \sim 2 RMA^2$$

where the relations (2.18) and the rigorous independence of different simultaneous q's have been used. Inserting this estimate of $\bar{\tau}$ into equation (5.6), and estimating $\int_0^\infty R_{\beta\beta}(\tau) R_{\gamma\gamma}(\tau)\, d\tau \sim \frac{1}{2} R^2 \bar{\tau}$, one finds

$$(5.8) \qquad \bar{\tau}\, \dot{E} \sim \bar{\tau}^2\, NMA^2\, R^2 \sim \frac{1}{2} NR \sim E$$

for the change of energy in one characteristic period. Thus the violation of energy

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*Consider, for example, the system of just three modes with $A_{123} = 1$, $A_{231} = -\frac{1}{2}$, $A_{312} = -\frac{1}{2}$. Its equations of motion are invariant under exchange of q_2 and q_3.

conversation is gross*.

The inviscid system just described may be considered a model of the equilibrium statistical mechanics of the inertial range of high Reynolds number turbulence. It is hard to see how the transition from equilibrium to non-equilibrium statistical mechanics (non-equipartition), or a more careful accounting of the detailed structure of the A's, could alter the estimate (5.8) by a very large factor. The result obtained then suggests rather strongly that if the hypothesis of quasi-normal two-point, two-time distribution is applied to actual stationary turbulence, the net transfer of energy to modes in the inertial range indicated by the sum over integrals in equation (4.4) represents a total rate of input $\dot{E}_i \sim E_i/\tau_i$, where E_i is the energy contained in the inertial range and τ_i is the order of some characteristic time in this range. The viscous dissipation in this range is known to be very much less for high Reynolds number.

It may be of interest, in concluding this section, to outline directly in terms of the field $\vec{u}(\vec{x},t)$ an equivalent of the argument leading to (5.6). The analog of equation (3.3) for the altered non-dissipative system is

$$\dot{C}_{ik}(\vec{y},0) = \int_0^\infty \left[-\eta^2(K^{-2}\nabla^2)P_{ij}(\vec{\nabla})P_{km}(\vec{\nabla})\nabla_\ell\nabla_n\right]E_{j\ell mn}(\vec{y},-\tau)\,d\tau. \tag{5.9}$$

Contracting with respect to i, k, and setting $\vec{y} = 0$, one obtains on the left side $\langle \dot{u}_i\dot{u}_i\rangle$, which is proportional to the mean rate of increase of energy. Under the two point, two-time quasi-normality hypothesis (3.13) one then obtains

$$\langle \dot{u}_i u_i\rangle = \int_0^\infty \Big\{ -\eta^2(K^{-2}\nabla^2)P_{jm}(\vec{\nabla})\nabla_\ell\nabla_n \times \left[C_{mj}(-\vec{y},\tau)C_{n\ell}(-\vec{y},\tau) + C_{nj}(-\vec{y},\tau)C_{m\ell}(-\vec{y},\tau)\right]\Big\}_{\vec{y}=0} d\tau, \tag{5.10}$$

where the identity $P_{ij}P_{im} = P_{jm}$ has been used. The proof that the right side of (5.10) is non-negative may be shown to follow from the fact that products of tensor correlation functions are themselves correlation functions and from the fact that $\eta^2(K^{-2}\nabla^2)P_{jm}(\nabla)$, $-\nabla_\ell\nabla_n$ are positive definite Hermitian tensor operators. Let us

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* It may be noted that even if the moments $\int_0^\infty R_{\beta\beta}(\tau)\,d\tau$ were to vanish, the quantities $\int_0^\infty R_{\beta\beta}(\tau)R_{\gamma\gamma}(\tau)d\tau$ should not be less than $R^2\bar{\tau}$ by a large factor, since the mean value of $|R_{\beta\beta}(\tau)|$ in the negative region should certainly be less than the mean value in the positive region nearer the origin.

denote the whole integrand in (5.10) by $Y(\tau)$; it is apparent that by squaring (5.1) and averaging over the distribution we obtain

$$(5.11) \qquad \langle \dot{u}_i \dot{u}_i \rangle = Y(0).$$

This, of course, cannot vanish, except trivally, so that if the dependence of $Y(\tau)$ on τ is of the sort supposed above for $R_{\beta\beta}(\tau)$, the right side of (5.10) does not vanish. The estimate (5.8) is readily corroborated from (5.10) and (5.11).

6. Conclusions

The investigation carried out in Sections 4 and 5 suggests strongly that the two-time, two-point quasi-normality hypothesis, applied to high Reynolds number turbulence, leads to a gross violation of energy conservation. Evidently, normal relation of second and fourth moments of the two-time, two-point distribution does not permit those distribution-averaged phase relations among coupled triads of modes which are required to maintain mean energy balance in all the elementary three-mode interactions*.

The consequences would appear to be serious with respect to the foundations of Chandrasekhar's recent turbulence theory[6], [7], which is based on this hypothesis. However, this does not in itself imply that the results of Chandrasekhar's theory are necessarily badly in error. The consequences of the quasi-normality hypothesis depend very strongly on the particular way in which it is employed. If in (3.7) one substitutes an integral expression of the form (2.20) for q_β or q_γ instead of q_α it is readily verified that an essentially <u>negative</u> result for $\Omega_{\alpha\beta\gamma}$ is obtained. Thus, a suitably symmetric application of the quasi-normality hypothesis could result in expressions for the energy transfer which are less grossly in error than equation (4.4)**. *A priori*, it is not implausible that Chandrasekhar's treatment

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*The set of moments defining any realizable statistical distribution must obey a set of generalized Schwarz inequalities. In the case at hand, if second and third moments of the two-time distribution of the q's are related by tha equations of motion, it is impossible, under the quasi-normality hypothesis, to satisfy non-trivially certain of these inequalities relating second, third, and fourth moments. In effect, the correlations between q_α and $q_\beta q_\gamma$ and between $q_\mu q_\lambda$ ($A_{\alpha\beta\gamma}, A_{\alpha\mu\lambda} \neq 0$; $\beta, \gamma \neq \mu, \lambda$) required by the equations of motion induce a correlation between $q_\beta q_\gamma$ and $q_\mu q_\lambda$ so that the moment $\langle q_\beta q_\gamma q'_\mu q'_\lambda \rangle$ does not vanish, as it would in a quasi-normal distribution.

**Writing, for the idealized dissipationless system,

$$\Omega_{\alpha\beta\gamma} = \frac{2}{3} A_{\alpha\beta\gamma} \left[\langle q_\alpha q_\beta q_\gamma \rangle + \langle q_\alpha q_\beta q_\gamma \rangle + \langle q_\alpha q_\beta q_\gamma \rangle \right],$$

and substituting integral expressions for q_α, q_β, q_γ respectively in the three terms, equation (3.11) gives $\Omega_{\alpha\beta\gamma} = 0$ if $R_{\alpha\alpha}(\tau) = R_{\beta\beta}(\tau) = R_{\gamma\gamma}(\tau)$.

in terms of the defining scalars of isotropic turbulence accomplishes this. It may not be concluded, therefore, from the evidence presented, that Chandrasekhar's theory necessarily leads to seriously incorrect physical results. What does seem evident is that the two-point, two-time quasi-normality hypothesis, which, to start, is highly arbitrary, must be considered a most suspect theoretical foundation if only a particular choice of the method of its application gives the possibility of avoiding grossly unphysical results.

The conclusions reached regarding the two-point, two-time quasi-normality hypothesis do not in themselves indicate the impossibility that the simultaneous distribution may obey a quasi-normality hypothesis. Some insight into this question perhaps is provided by the dissipationless system discussed in Section 5. For this system it was seen that the simultaneous distribution could be rigorously normal. However, the third moments of the corresponding two-time distribution do not vanish, nor are the fourth moments of the two-time distribution normally related to second moments. This may be seen directly by expressing the derivatives $\partial S_{\mu\lambda\sigma}/\partial\tau$, $\partial^2 T_{\alpha\mu\lambda\sigma}/\partial\tau^2$ at $\tau = 0$ in terms of the simultaneous distribution, using the equation of motion (5.3), and verifying that they do not all vanish for $\alpha,\mu,\lambda,\sigma$ all different. If, now, viscosity and driving forces are reintroduced, the third moments of the simultaneous distribution take on non-vanishing values. It seems at least plausible that, like the vanishing of the third moments, the normality relation between fourth and second moments in the simultaneous distribution is a property peculiar to equipartition and may not be realizable when there is a strong mean energy transfer, if the second, third, and fourth moments are to satisfy the conditions for the existence of the distribution function.

In conclusion, it should be recalled that the considerations of the present paper were confined to examination of the energy balance in modes for which the external driving forces were assumed to vanish. The results obtained do not establish the impossibility of an approximately quasi-normal distribution for modes in the energy containing range, where the external forces cannot be ignored in stationary turbulence.

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